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«PERFECT JOHNSON S-ACTS AND THEIR FRAGMENTS»

ABSTRACT
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The relevance of the topic. Model theory is one of the key areas of fundamental mathematics and attracts significant interest from modern researchers. Recent advancements demonstrate positive trends in its development, confirming that this discipline will play an essential role in the future of mathematical sciences. It is worth noting that classical model theory introduces concepts primarily based on the study of complete theories.

Jonsson theories form a subclass of inductive theories and are inherently incomplete. However, they encompass a wide range of classical algebraic structures, including groups, abelian groups, fields of fixed characteristic, Boolean algebras, S-acts, and more. The eminent mathematician and logician H. J. Keisler, in his review article "Fundamentals of Model Theory" from the four-volume "Handbook of Mathematical Logic" (edited by J. Barwise), defined foundational concepts and directions for the development of model theory. Keisler identified two historical trends in model theory's evolution, referred to as the "Western" and "Eastern" schools of thought. This division stems from the geographical locations of A. Tarski (on the U.S. West Coast from 1940) and A. Robinson (on the East Coast from 1967 until his sudden death in 1975). While the geographic relevance has faded, this distinction remains mathematically significant.

The "Western" model theory developed under the traditions of Skolem and Tarski, focusing on number theory, mathematical analysis, and set theory. It employed the full range of first-order logic formulas. The "Eastern" model theory, rooted in the traditions of Robinson, concentrated on problems in abstract algebra, often using formulas with two quantifier sets. It paid particular attention to quantifier-free and existential formulas.

The study of the Jonsson theories started from the works of Jonsson, as well as the works of Morley, and Vaught. In the mid-1980s, T.G. Mustafin introduced a novel approach to the study of Jonsson theories. He identified a natural subclass of Jonsson theories, which he termed "perfect Jonsson theories". His method focused on analyzing properties of arbitrary Jonsson theories through the transfer of central extension properties. However, his approach required an additional axiom involving the existence of a strongly inaccessible cardinal in the Zermelo-Fraenkel set theory framework. During the "5th Kazakh-French Model Theory Colloquium," renowned model theory experts Ye.A. Palyutin and B. Poizat suggested revising this definition. Ye.T. Mustafin later redefined κ-homogeneity

and the semantic model, leading to a modified definition of the perfection of Jonsson theories. Results obtained earlier were revisited and presented within the framework of the new definition.

The studies discussed above align with the "Eastern" model theory tradition. T. G. Mustafin described generalized Jonsson theories for Boolean algebras. Subsequent research on Jonsson theories identified new classes of positive Jonsson theories.

The study of any algebraic system is closely connected to the study of its elementary theory. Since we operate within the framework of Jonsson theories, which are generally incomplete, it is necessary first to revisit the results obtained for the complete theory of unars. Recall that an unar is a structure with a signature containing a single unary functional symbol.

Yu.E. Shishmarev made foundational contributions to this field. In 1972, he proved three theorems related to the complete theory of unars with infinite models. He defined the conditions required for a limited theory to achieve categoricity in countable and uncountable cardinalities and for an unlimited theory to achieve categoricity in uncountable cardinalities. A.A. Ivanov concluded that the elementary theory of unars is decidable. He also obtained results on strongly ultra-homogeneous unars. These findings are related to defining criteria for allowing quantifier elimination in complete theories of unars, as well as the fact that every complete limited or unlimited theory of unars with infinite models is not finitely axiomatizable. A.N. Ryaskin calculated the number of models for complete theories of unars and, in one of his works, studied the finite cover properties of complete theories of unars.

From English-language sources, the works of Leo Marcus are of particular interest. Specifically, his study of the criteria for the case where the model M of the language L_1 is minimal and consists of simple or non-simple components. The language L_1 consists of a unary functional symbol and a predicate symbol (equality). Marcus achieved significant results regarding relationships between the components, including equivalence conditions and the criteria for disjoint union.

The study of Jonsson theories for unars began with the works of T.G. Mustafin. The semantic model's characterization of unars was developed by A.R. Yeshkeyev and T.G. Mustafin, who also demonstrated that the Jonsson theory of unars is perfect.

It is worth noting that describing a class of specific algebraic systems defined within a particular model-theoretic language may not always result in a characterization in that language.

On the other hand, it is important to emphasize that the theory of all unars corresponds to the Jonsson theory of S-acts over a cyclic monoid. Specifically, one can examine models of such a theory in the form of an algebraic system $\{M;\ f_\alpha\}_{\alpha\in M}$, where $M \times M \to M$, and M is a cyclic monoid satisfying the

following properties: $f_e(a) = a$ for $e \in$ M and all $a \in M$; $f_{\alpha\beta}(a) = f_\alpha\left(f_\beta(a)\right)$ for all $a \in M$ and all $\alpha, \beta \in M$.

A monoid with a single generating element is called cyclic. Any cyclic monoid either behaves like a cyclic group or is obtained by externally adjoining an identity element to a cyclic semigroup..

The study of S-acts over monoids and groups, prior to this dissertation, was primarily conducted within the framework of their complete theories. As mentioned earlier, Jonsson theories are generally incomplete. While the descriptions of unars and their theories in complete settings are well-documented, the incompleteness of the theory of all unars is the only known fact regarding their incomplete theories. Since the study of Jonsson theories refines and generalizes existing results in classical model theory, the relevance of this research is beyond question.

The purpose of the work. The primary objective of this dissertation is to study the model-theoretic properties of Jonsson theories of S-acts as well as the classes of their models. By Jonsson theories of S-acts, we mean: Existentially positive Mustafin theories of S-acts over a group; Robinson theories of universal unars and Robinson theories of undirected graphs; Jonsson theories of primitives (∀∃-consequences) of unars in a new signature expanded by a unary predicate symbol, distinguishing the existentially closed model, and by a constant symbols.

Research objectives.

1. To find the criterion for the cosemanticness of semantic models of classes of Jonsson $\exists PM$-theories of S-acts over a group for the set $PSp(K_\Pi)/\bowtie_{\exists PM}$.

2. To describe the cosemanticness classes of the set $RSp(J\mathbb{C}_U)/_\bowtie$ (the Robinson spectrum of the semantic Jonsson quasivariety of Robinson unars) and the corresponding characteristics of their semantic models $\mathfrak{C}_\Delta \in J\mathbb{C}_U$.

3. To describe the cosemanticness classes of the set $RSp(J\mathbb{C}_U)/_\bowtie$ (the Robinson spectrum of the semantic Jonsson quasivariety of Robinson unars) for ω-categorical Robinson theories of unars.

4. To describe the Jonsson primitives of unars in a signature expanded by a new unary predicate symbol P^1, distinguishing the existentially closed model in the semantic model of the Jonsson primitive of unars, and by a constant symbols.

The object of research. Perfect Jonsson theories of S-acts over a group, perfect Jonsson theories of S-acts over a cyclic monoid, and the classes of their models.

Research methods. The research employed a semantic method characterized by the transfer of properties of center of a Jonsson theory to itself. Additionally, methodologies from classical model theory and universal algebra were applied.

Main provision. The following new scientific results were obtained in the study:

1. The criterion for the cosemanticness of semantic models of classes of Jonsson $\exists PM$-theories of S-acts over a group for the set $PSp(K_{\Pi})/\bowtie_{\exists PM}$ (theorem 1.4.6).

2. The description of the cosemanticness classes of the set $RSp(J\mathbb{C}_U)/_{\bowtie}$ (the Robinson spectrum of the semantic Jonsson quasivariety of Robinson unars) and the corresponding characteristics of their semantic models $\mathfrak{C}_{\Delta} \in J\mathbb{C}_U$ (theorem 2.2.1).

3. The description of the cosemanticness classes of the set $RSp(J\mathbb{C}_U)/_{\bowtie}$ (the Robinson spectrum of the semantic Jonsson quasivariety of Robinson unars) for ω-categorical Robinson theories of unars (theorem 2.3.7).

4. The description of the Jonsson primitives of unars in a signature expanded by a new unary predicate symbol P^1, distinguishing the existentially closed model in the semantic model of the Jonsson primitive of unars, and by a constant symbols (theorem 2.5.8).

Description of the main results of the study. The first section of the dissertation outlines the fundamental definitions and theorems of classical model theory, as well as research findings related to the concept of the Jonsson spectrum for a fixed class of models with the signature of S-acts, where the group is assumed to act as a monoid in the S-acts. The concept of the Jonsson spectrum is effective when describing the model-theoretic properties of classes of algebras whose theories have the joint embedding and amalgamation properties. Typically, it suffices to consider universal-existential statements that hold true in the models of this class. Before the information presented in the first section of this dissertation, the Jonsson spectrum dealt exclusively with Jonsson theories. However, studying this concept has enabled the definition of a positive Jonsson spectrum, whose elements may include non-Jonsson theories. This arises because the definition of existentially positive Mustafin theories, as discussed in this section, involves not only isomorphic embeddings but also immersions. Consequently, the properties of joint embedding and amalgamation in these definitions rely on immersions. As a result, the derived theories are not necessarily Jonsson theories. It is noteworthy that this approach to studying the Jonsson spectrum is justified, as even in the case of non-Jonsson theories, there exists a regular method for identifying a Jonsson theory that satisfies the previously known concepts and results. However, this also directly applies to the existentially positive Mustafin theory under consideration.

The second section of the dissertation is devoted to the study of the semantic Jonsson quasivariety of universal unars, i.e., structures of a signature containing a single unary functional symbol, and undirected graphs with a signature containing a binary predicate symbol. New concepts of the semantic Jonsson quasivariety of Robinson unars $J\mathbb{C}_U$ and its elementary theory were defined. To prove the main result of the study, the Robinson spectrum $RSp(J\mathbb{C}_{\mathfrak{U}})$ and its partition into equivalence classes $[T_{\mathfrak{U}}]$ under the relation of cosemanticness were

studied. The characteristics of such equivalence classes $[T_{\mathfrak{U}}] \in RSp(J\mathbb{C}_{\mathfrak{U}})$ were analyzed in detail.

The main result is the following existence theorem: of the characteristics of each class $[T_{\mathfrak{U}}]$, consisting of Robinson theories of unars; of the characteristics of the class $[T_{\mathfrak{U}}]$ for any arbitrary characteristic; of the criterion for equivalence between two classes $[T_{\mathfrak{U}_1}]$, and $[T_{\mathfrak{U}_2}]$. Additionally, the Robinson spectra $RSp(J\mathbb{C}_{\mathfrak{U}})$ (the Robinson spectrum of the semantic Jonsson quasivariety of Robinson unars) and $RSp(J\mathbb{C}_{\mathfrak{G}})$ (the Robinson spectrum of the semantic Jonsson quasivariety of Robinson unoriented graphs) were examined, along with their partition into equivalence classes $[T_{\mathfrak{U}}]$ and $[T_{\mathfrak{G}}]$ under the cosemanticness. The key results of this study lead to the following useful corollaries: countably categorical Robinson theories of unars are totally categorical; countably categorical Robinson theories of undirected graphs are totally categorical. The second section also considers the theory of all unars and the class of existentially closed models of this theory. An expanded signature of unars was analyzed by introducing a new unary predicate symbol that designates the existentially closed model in the semantic model of the Jonsson theory of unars, along with a new constant symbol. Several results concerning universals and primitives of existentially closed Jonsson unars in the considered theory were obtained.

Substantiation of the novelty and importance of the results obtained. As outlined above, the problem of describing Jonsson theories of algebraic systems, which constitutes the foundational task of this dissertation, is highly relevant. In addressing this problem, new concepts such as the Jonsson spectrum, specifically the Robinson spectrum, and the semantic Jonsson quasivariety were utilized. Moreover, the main result of the first section of the study connects both to Jonsson theories of the positive Jonsson spectrum and to non-Jonsson theories. Such cases have not been previously studied. The work is theoretical in nature, and its practical significance is assessed through the applied value of model theory. This research is fundamental and makes a significant scientific contribution to the advancement of mathematical sciences.

Approbation of the work. The main results of the dissertation were presented and discussed at the following international conferences and scientific seminars related to the dissertation topic:

- Presentation on "Existentially Positive Mustafin Theories of S-Acts over a Group" at the traditional international April conference dedicated to the Day of Science Workers of the Republic of Kazakhstan (Institute of Mathematics and Mathematical Modeling, Almaty, April 6–8, 2022).

- Presentation on "The Perfect Jonsson S-acts" at the international scientific conference "Current Issues of Mathematics, Mechanics, and Informatics" dedicated to the 80th anniversary of Professor T.G. Mustafin (Buketov Karaganda University, Karaganda, September 8–9, 2022).

- Presentation on "The Positive Jonsson Spectrum of $\exists PM$-Theories of S-Acts over a Group" at the 3rd International Taimanov Readings "Modern Mathematics: Problems and Applications," dedicated to the 85th anniversary of Korkyt Ata Kyzylorda University (Korkyt Ata Kyzylorda University, Kyzylorda, November 25, 2022).

- Presentation on "On Countable Categoricity of Semantic Jonsson Quasivarieties of Unars and Graphs" at the traditional international April conference dedicated to the Day of Science Workers of the Republic of Kazakhstan (Institute of Mathematics and Mathematical Modeling, Almaty, April 5–7, 2023).

- Presentation on "On Robinson Spectrum of the Semantic Jonsson Quasivariety of Unars" at the traditional international April conference dedicated to the Day of Science Workers of the Republic of Kazakhstan (Institute of Mathematics and Mathematical Modeling, Almaty, April 5–7, 2023).

- Presentation on "The Robinson Spectrum of the Semantic Jonsson Quasivariety of Unars" at the "Model Theory" scientific seminar named after E.A. Palyutin, Sobolev Institute of Mathematics (Novosibirsk, Russia), jointly with the Institute of Mathematics and Mathematical Modeling (Almaty, Kazakhstan), May 10, 2023.

- Presentation on "On Semantic Jonsson Quasivariety of Robinson Unars" at the 7th World Congress of Mathematicians of the Turkic World (TWMS Congress-2023) (Turkistan, September 20–23, 2023).

- Presentation on "On Semantic Jonsson Quasivariety of Undirected Graphs" at the 7th World Congress of Mathematicians of the Turkic World (TWMS Congress-2023) (Turkistan, September 20–23, 2023).

- Presentation on "On Δ-Jonsson Spectrum of Δ-PJ-Theories" at the traditional international April conference dedicated to the Day of Science Workers of the Republic of Kazakhstan (Institute of Mathematics and Mathematical Modeling, Faculty of Mechanics and Mathematics, Al-Farabi Kazakh National University, Almaty, April 5–7, 2024).

- Presentations on "On the Quantity of Equivalence Classes of Robinson Spectrum of Unars" and "On the Characteristics of Equivalence Classes of Robinson Spectrum Regarding Their Primitive" at the traditional international April conference dedicated to the Day of Science Workers of the Republic of Kazakhstan (Institute of Mathematics and Mathematical Modeling, Faculty of Mechanics and Mathematics, Al-Farabi Kazakh National University, Almaty, April 5–7, 2024).

- Presentation during the 16th International Summer School-Conference "Frontier Questions of Model Theory and Universal Algebra" (Sobolev Institute of Mathematics, July 8–13, 2024).

Compliance with the directions of scientific development or governmets programs. The dissertation research topic aligns with the priority area of development in "Natural Sciences", within the specialized scientific direction

"Fundamental and Applied Research in Mathematics, Mechanics, Astronomy, Physics, Chemistry, Biology, Informatics, and Geography". The work was carried out as part of grant funding for scientific and scientific-technical projects by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan under the following themes:

1. "The cosemantic classes and their classes of models", IRN AR09260237, for 2021–2023.

2. "Fragments of definable subsets of the semantic model of fixed Jonsson theory", IRN AP23489523, for 2024–2026.

Publications. The main results of the dissertation research have been published in 14 works, including articles in journals indexed in the Scopus database (46th percentile), a journal recommended by the Committee for Quality Assurance in Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan, and proceedings of international scientific conferences.

Description of the doctoral student's contribution to the preparation of each publication. The main results of the dissertation have been published in 4 works: 3 articles in journals indexed in the Scopus database (46th percentile) and 1 article in a journal recommended by the Committee for Quality Assurance in Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan. The doctoral candidate's contribution to each publication includes analyzing the relevant literature, formulating the main and auxiliary results, and proving these results.

The structure and scope of the dissertation. The dissertation consists of 62 pages and includes the following structural elements: an introduction, two sections, a conclusion, and a list of references.

The number of sources used is 57.

Keywords. Jonsson theory, semantic model, perfect Jonsson theory, S-act, S-act over a cyclic monoid, S-act over a group, graph, undirected graph, primitives, universals, positivity, Jonsson spectrum, Robinson spectrum, semantic Jonsson quasivariety.